Optimal transport and applications

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The optimal transport problem first formulated by Gaspard Monge in 1781 consists in finding the cheapest way to transport a distribution of mass to another. In the 1940's, Leonid Kantorovich introduced a powerful relaxation of the problem and in his seminal work in the late 1980's, Yann Brenier solved the problem for the quadratic cost, stimulating an intensive stream of research with an incredible range of (sometimes unexpected) applications. The two Fields medallists Cédric Villani (who wrote reference textbooks on the topic) and Alessio Figalli made fundamental contributions to the field, which has gained maturity since Brenier's pioneering works and is now well-established.

The aim of this course is to present some of the main results and methods for the rich theory originating from the problem of Monge and Kantorovich, the entropic version of the problem and its connection with the popular Sinkhorn algorithm will also be studied. The course is intended to be self-contained but requires some familiarity with classical functional analysis and measure theory. It will consist of 14 lectures of 1,5 hour each, as detailed below.

Course 1

Monge's formulation of the optimal transport problem, transport maps. The existence problem: existence of transport maps for nonatomic sources, examples of non existence of optimal transport maps.

Course 2

Construction of transport maps. Monotone transport, Knothe-Rosenblatt transport. Transport by a flow, the continuity equation and Dacorogna-Moser transport.

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Course 3

Monge-Kantorovich formulation of the optimal transport problem, transport plans, disintegration. Existence of optimal plans. Relaxation is exact for nonatomic sources.

Course 4

Two special cases: the assignment problem, Birkhoff-von Neumann Theorem, the unidimensional case and optimality of the monotone transport for submodular costs.

Course 5

Kantorovich duality. Heuristic derivation, rigorous proof by the Fenchel-Rockafellar Theorem, existence of optimal potentials, primal-dual relations, characterization of optimal plans.

Course 6

An alternative viewpoint: cyclical monotonicity, c-concavity, c-subdifferentiability, cyclical monotonicity as an optimality condition, Rockafellar's Theorem and generalizations.

Course 7

Twisted costs, existence of optimal transport maps. Quadratic case, Brenier's theorem, link with Monge-Ampère equations. Applications.

Course 8

Dynamical formulation, Benamou-Brenier formula.

Course 9

Wasserstein distances W_p, Wasserstein spaces, proof of the triangle inequality by the gluing Lemma. The special case of W_1 on $\mathbf{R}^d$, Beckmann's minimal flow formulation.

Course 10

Displacement convexity, McCann's interpolation, geodesics, displacement convex functionals, applications, functional inequalities.

Course 11

Wasserstein gradient flows, JKO scheme. Detailed analysis of the Fokker-Planck case.

Course 12

Entropic optimal transport, duality, optimality conditions, potentials, Sinkhorn algorithm and its convergence.

Course 13

Entropic optimal transport, quantitative convergence, linear convergence of Sinkhorn.

Course 14

An optimal multi-marginal optimal transport problem: Wasserstein barycenters.